Scientific report

1 Scientific description

In its 2023 stage, financing contract no. 30/2021 aiming to obtain new geometric and topological properties of locally conformal Kähler manifolds, in accordance with the stated objectives, namely:

O1. Investigation of LCK submersions and products;

O2. LCK structures on solvmanifolds, coexistence with other special metrics;

O3. Variational problems in LCK geometry.

To achieve the assumed objectives, 9 scientific papers were elaborated during this period, 4 among them being accepted or published in ISI-ranked journals and 5 papers are under review. Moreover, 7 papers elaborated and submitted for evaluation for a possible publication in the previous stages were published or accepted for publication in ISI-ranked journals, as follows.

- L. Ornea, M. Verbitsky, *Compact homogeneous locally conformally Kähler manifolds are Vaisman. A new proof* - published in **Rivista di Matematica dell'Università di Parma**, Volume 13 (2022) no. 2, 439-448.
- D. Angella, M. Parton, V. Vuletescu, *On locally conformally Kähler threefolds with algebraic dimension two* - published in **International Mathematics Research Notices**, Volume 2023, Issue 5, March 2023, Pages 3948–3969.
- V. Slesar, G.-E. Vîlcu, *Vaisman manifolds and transversally Kähler-Einstein metrics* - published in **Annali di Matematica Pura ed Applicata** 202 (2023), no. 4, 1855–1876.
- L. Ornea, M. Verbitsky, V. Vuletescu, *Do products of compact complex manifolds admit LCK metrics?* - accepted for publication in **Bulletin of the London Mathematical Society**; `http://doi.org/10.1112/blms.12962`.
- L. Ornea, M. Verbitsky, *Mall bundles and flat connections on Hopf manifolds* - accepted for publication in **Annales de l'Institut Fourier**.
- L. Ornea, M. Verbitsky, *Non-linear Hopf manifolds are locally conformally Kähler* - accepted for publication in **Journal of Geometric Analysis**; doi:10.1007/s12220-023-01273-2.

- C. Gherghe, G.-E. Vîlcu, *Harmonic maps on locally conformal almost cosymplectic manifolds* - accepted for publication in **Communications in Contemporary Mathematics**; doi:10.1142/S0219199723500529.

The content of these 9 papers realized in 2023, that cover completely the objectives proposed for this period, can be synthesized as follows:

A1. L. Ornea, M. Verbitsky, *Bimeromorphic geometry of LCK manifolds*, **Proceedings Amer. Math. Soc.** (2023), accepted.

For LCK manifolds M with exact Kaehler form on the minimal cover, we prove that any bimeromorphic map $M \to M'$ is in fact holomorphic; in other words, M has a unique minimal model. This can be applied to a wide class of LCK manifolds, such as the Hopf manifolds, their complex submanifolds and to OT manifolds.

A2. N. Bin Turki, U.C. De, A.A. Syied, G.-E. Vîlcu, *Investigation of space-times through W_2-curvature tensor in $f(R,G)$ gravity*, **Journal of Geometry and Physics** 194 (2023), 104987.

The W_2-curvature tensor is an important geometric invariant with relativistic significance, introduced in the early 1970s by Pokhariyal and Mishra, which can be identified in class 4 in the classification of skew-symmetric operators. In this work, we investigate 4-dimensional space-times admitting W_2-curvature tensor in $f(R,\mathcal{G})$ modified theory of gravity. It is proved that the W_2-curvature flat perfect fluid space-times obeying $f(R,\mathcal{G})$ gravity exhibit inflation. Also, it is shown that the isotropic pressure and the energy density of such space-times are constant. It is to be noted that in such space-times the considered energy conditions are consistently satisfied if the scalar curvature is positive. Next, we study perfect fluid space-times admitting divergence free W_2-curvature tensor in $f(R,\mathcal{G})$ gravity. Amongst other results, we prove that if the energy-momentum tensor of such space-times is bi-recurrent, then either the space-times exhibit inflation or their isotropic pressure and energy density are constants.

A3. M. Aquib, M.S. Lone, C. Neacşu, G.-E. Vîlcu, *On δ-Casorati curvature invariants of Lagrangian submanifolds in quaternionic Kähler manifolds of constant q-sectional curvature*, **Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemática** 117 (2023), Article number: 107.

Lagrangian submanifolds, a class of Riemannian submanifolds that arose naturally in the context of Hamiltonian mechanics, play an important role in some modern theories of physics. In this paper, using an optimization technique on submanifolds immersed in Riemannian manifolds, we first obtain some sharp inequalities for δ-Casorati curvature invariants of Lagrangian submanifolds in quaternionic space forms, i.e. quaternionic Kähler manifolds of constant q-sectional curvature. Then we show that in the class of Lagrangian submanifolds in quaternionic space forms, there are only two subclasses of ideal Casorati submanifolds, namely the family of totally geodesic submanifolds and a particular subfamily of H-umbilical submanifolds. Finally, we provide some examples to illustrate the obtained results. In particular, we point out that an entire family of ideal Casorati Lagrangian submanifolds can be constructed using the concept of quaternionic extensor introduced by Oh and Kang in the early 2000s.

A4. B.-Y. Chen, M.A. Lone, A.-D. Vîlcu, G.-E. Vîlcu, **Curvature properties of spacelike hypersurfaces in a RW spacetime**, Journal of Geometry and Physics 194 (2023), 105015.

Curvature invariants of both intrinsic and extrinsic nature play a significant role in elucidating the geometry of a spacetime. In particular, these invariants are useful in detecting event horizon of black holes. Notable examples of spacetimes are provided by the generalized Robertson-Walker (GRW) models. An $(m+1)$-dimensional GRW spacetime is a Lorentzian warped product $I \times_f N$ endowed with the warped metric $\hat{g} = -dt^2 + f^2(t)g$, having as base an open real interval I equipped with the metric $-dt^2$ and as fiber any Riemannian space (N, g) of dimension m, where f is a smooth positive-valued function on I. In this article, we focus our study on the GRW spacetimes having the fiber a Riemannian space with metric $g = g_k$ of constant sectional curvature k and denoted by $L_1^{m+1}(k, f)$, i.e. RW spacetimes. We obtain in this work the lower bound of the (generalized) normalized Casorati curvatures for a spacelike hypersurface $\mathcal{H}$ in the GRW spacetime $L_1^{m+1}(k, f)$ in terms of the (normalized) scalar curvature of $\mathcal{H}$, the constant sectional curvature k of the fiber, the normal hyperbolic angle of the hypersurface and the warping function f. We also derive the conditions under which this lower bound is reached. We finally apply the results to some basic cosmological models, namely Lorentz-Minkowski, de Sitter, Einstein-de Sitter, anti de Sitter and steady state spacetimes.

A5. M. Aprodu, *V-minimal submanifolds*, arXiv:2306.02104.

We introduce the notion of V-minimality, for V a smooth vector field on a Riemannian manifold. This is a natural extension of the classical notion of minimality which is achieved for V=0. Complex submanifolds in a locally conformal Kähler manifold are V-minimal, for V a suitable integer multiple of the Lee vector field. To emphasis the utility of this notion we extend some previous results. Specifically, we prove that a PHH submersion is V-harmonic if and only if it has minimal fibres and a PHH V-harmonic submersion pulls back complex submanifolds to V minimal submanifolds.

A6. S. Dăscălescu, *Finiteness conditions for Hopf superalgebras*, arXiv:2202.12398.

The theory of integrals for Hopf algebras was initiated by Larson and Sweedler in the 1960's, and continued with Sullivan's proof of the uniqueness of integrals in early 1970's. Integrals have been of great use in understanding the structure of Hopf algebras. Their study was in close relationship to finiteness properties of the underlying coalgebra structure of the Hopf algebra. In the case of Hopf superalgebras, a systematic study of integrals was initiated in by Scheunert and Zhang in 2001, and it has been related to integration on Lie supergroups. It is natural to ask whether the existence of integrals is also related to finiteness coalgebra properties in the Hopf superalgebra case. We answer this in

Theorem A. *Let A be a Hopf superalgebra. Then the following are equivalent.*
(i) A has non-zero left integrals.
(ii) A is left co-Frobenius as a coalgebra.
(iii) A is left quasi-co-Frobenius as a coalgebra.

(iv) A is a left semiperfect coalgebra.
Moreover, these conditions are also equivalent to their right hand side versions.

A standard method in the study of Hopf superalgebras is the reduction to usual Hopf algebras by applying a process of bosonization. More precisely if A is a Hopf superalgebra, then there is an action and a coaction of the group Hopf algebra $k\mathbb{Z}_2$ of the cyclic group of order 2 on A, and then the tensor product $A \otimes k\mathbb{Z}_2$ has an algebra structure (a smash product) and a coalgebra structure (a smash coproduct), which are compatible and make $A \otimes k\mathbb{Z}_2$ a Hopf algebra. Our approach for proving Theorem A uses bosonization, so we need to understand how finiteness properties transfer between A and its bosonization. In fact we prove a much more general transfer result, which is of interest by itself.

Theorem B. *Let C be a right comodule coalgebra over a finite dimensional Hopf algebra H, and let $C \rtimes H$ be the associated smash coproduct. The following assertions hold.*
(1) *$C \rtimes H$ is left quasi-co-Frobenius if and only if so is C.*
(2) *$C \rtimes H$ is left co-Frobenius if and only if so is C.*

A7. V. Marchidanu, *Complex structures on the product of two Sasakian manifolds*, arXiv:2307.04609.

A Sasakian manifold is a Riemannian manifold whose metric cone admits a certain Kähler structure which behaves well under homotheties. We show that the product of two compact Sasakian manifolds admits a family of complex structures indexed by a complex nonreal parameter, none of whose members admits any compatible locally conformally Kähler metrics if both Sasakian manifolds are of dimension greater than 1. We compare this family with another family of complex structures which has been studied in the literature. We compute the Dolbeault cohomology groups of these products of compact Sasakian manifolds.

A8. O. Preda, M. Stanciu, *Locally conformally Kähler spaces and proper open morphisms*, arXiv:2311.14372.

In this paper, we prove a stability result for the non-Kähler geometry of locally conformally Kähler (lcK) spaces with singularities. Specifically, we find sufficient conditions under which the image of an lcK space by a holomorphic mapping also admits lcK metrics, thus extending a result by Varouchas about Kähler spaces.

A9. M.E. Aydin, R. Lopez, G.-E. Vîlcu, *Classification of separable hypersurfaces with constant sectional curvature*, arXiv:2309.06025.

In this paper, we obtain the full classification of separable hypersurfaces of constant sectional curvature in n-dimensional Euclidean space, generalizing the results obtained in dimension 3 first by Gronwall [Ann. of Math. (2) 32, No. 2, 313-326 (1931) in a particular case, and then by Hasanis and López [Manuscripta Math. 166, 403-417 (2021)] in the general case.

2 Progress Summary

The 2023 stage was completed with the full fulfillment of the assumed objectives (O1-O3), realizing a number of 9 scientific articles (A1-A9). The dissemination of the research results was achieved by participating with invited presentations at international conferences and departmental talks, as follows.

2.1 Invited talks at international conferences

1. L. Ornea: Cohomology of Complex Manifolds and Special Structures - III, January 23-27 2023, Trento.

2. L. Ornea: A Complex Differential Geometry Meeting, Torino, 17-19 May.

3. L. Ornea: Sasakian manifolds, Riemannian foliations, and related structures, June 26-29, Krakow.

4. L. Ornea: International Conference in Applied and Pure Mathematics, Iaşi, 9-13 November, 2023.

5. M. Stanciu: The Tenth Congress of Romanian Mathematicians, Pitești, June 30 - July 5, 2023.

6. V. Marchidanu: Workshop for Young Researchers in Mathematics 12th Edition, Alexandru Ioan Cuza University of Iași, May 18 – May 19, 2023.

7. V. Marchidanu: International Conference on Applied and Pure Mathematics (ICAPM2023), Iași, November 9-12, 2023.

8. G.-E. Vîlcu: International Conference ICES2023, 6-8 February 2023, King Saud University, Riyadh (Saudi Arabia).

9. G.-E. Vîlcu: The 4th International Workshop Differential Geometric Structures and Applications, May 11-13, 2023, Haifa University, Haifa.

10. G.-E. Vîlcu: The First Sharjah International Conference on Mathematical Sciences, Sharjah, 6-8 November 2023.

11. A. Otiman: The Tenth Congress of Romanian Mathematicians, Pitești, June 30 - July 5, 2023.

2.2 Invited departmental talks

1. A. Otiman: *Cohomology and special metrics in complex geometry*, May 2023, Aarhus University.

From the above, we deduce the following result indicators:

- Articles published in ISI indexed journals: 3

- Articles accepted in ISI indexed journals: 1
- Articles under evaluation in ISI indexed journals: 5
- Presentations at conferences: 12

The mobilities settled from the 2023 stage budget were the following.

1. L. Ornea: 11.05-21.05.2023, mobility in Italy, at the University of Rome 3, for participating with an invited talk at the conference "A Complex Differential Geometry Meeting at UniTo".
2. M. Stanciu: 10.09-17.09.2023, mobility in Danemarca, at the Aarhus University, for scientific collaboration with the LCK geometry specialists in the department.
3. M. Stanciu: 30.06-05.07.2023, mobility at Pitești for participating with a talk at the conference "Tenth Congress of Romanian Mathematicians".
4. V. Marchidanu: 14.01–07.03.2023, mobility in Brazil, at the Instituto Nacional de Matemática Pura e Aplicada (IMPA) Rio de Janeiro, for scientific collaboration with Prof. Misha Verbitsky on topics of common interest from locally conformal Kähler geometry, as well as for attending the course "Metric Geometry" during the summer school.
5. V. Marchidanu: 08-12.11.2023, mobility at Iași for attending the "International Conference on Applied and Pure Mathematics (ICAPM2023)".
6. G.-E. Vîlcu: 11.05-13.05.2023, mobility in Israel, at University of Haifa, for participating with an invited talk at the conference "The 4th International Workshop Differential Geometric Structures and Applications".
7. G.-E. Vîlcu: 6.11-8.11.2023, mobility in United Arab Emirates, at University of Sharjah, for participating with an invited talk at the conference "The First Sharjah International Conference on Mathematical Sciences".
8. V. Slesar: 28.10-05.11.2023, mobility in Spain, at University Santiago de Compostela, for scientific collaboration with Prof. Jesus Alvarez Lopez on topics of common interest from locally conformal Kähler geometry.

3 Summary of the stage

In its 2023 stage, entitled *LCK and related structures on complex manifolds*, financing contract no. 30/2021 aiming to obtain new geometric and topological properties of locally conformal Kähler manifolds, in accordance with the stated objectives.

The activities carried out consisted in:

- documentation and information,
- analysis and conception of solutions for realization,

- elaboration of scientific papers and
- participation in scientific events for the dissemination of the obtained results.

All activities were 100% completed by the deadline and according to the budget allocated by the contract and the objectives were fully achieved.

The output was:

- A number of 9 scientific papers (which is three times the planned number).
- 12 invited talks at international conferences.

Project manager,

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