Scientific report

1 Scientific description

In its 2022 stage, financing contract no. 30/2021 aiming to obtain new geometric and topological properties of locally conformal Kähler manifolds, in accordance with the stated objectives, namely:

O1. investigation of the deformations of Vaisman structures that preserve the foliation,

O2. degeneracy of the GSS manifolds (i.e. manifolds admitting global spherical shells, also called Kato manifolds) and obtaining new examples of LCK manifolds,

O3. the study of LCK structures on singular analytic spaces and the analytical classification of manifolds of complex dimension 3.

To achieve the assumed objectives, 17 scientific papers were elaborated during this period, 8 among them being accepted or published in ISI-ranked journals, an article appeared in a special volume of the "*Contemporary Mathematics*" collection published by AMS, and 8 papers are under review. Moreover, 2 papers elaborated and submitted for evaluation for a possible publication in the 2021 stage were published or accepted for publication in ISI-ranked journals, as follows.

- T. Albu, S. Dăscălescu, *Free objects and coproducts in categories of posets and lattices* - published in **Communications in Algebra** 50 (2022), 3178-3187.
- O. Preda, M. Stanciu, *Vaisman theorem for LCK spaces* - accepted for publication in **Annali della Scuola Normale Superiore di Pisa**.

The content of these 16 papers, that cover completely the objectives proposed for this period, can be synthesized as follows:

A1. L. Ornea, V. Slesar: *Deformations of Vaisman manifolds*, **Differential Geometry and its Applications** 85 (2022), 101940.

We construct a deformation of the Vaisman structure in such a way that the canonical foliation is not affected and the deformation concerns only the transverse orthogonal complement and the transverse Kähler geometry. This type of deformation is related to the deformations of second type on Sasakian manifolds, as defined by Belgun. For our construction, we need a basic 1-form with certain natural properties and we indicate a way of producing such 1-forms. This procedure does not cover the whole space of deformations, still it gives a way of obtaining new Vaisman structures out of a given one. We end with a completely worked-out application of the method to a classical Hopf surface.

A2. L. Ornea, M. Verbitsky, *Lee classes on LCK manifolds with potential*], **Tohoku Mathematical Journal** (2022), accepted.

An LCK manifold is a complex manifold (M, I) equipped with a Hermitian form ω and a closed 1-form θ, called the Lee form, such that $d\omega = \theta \wedge \omega$. An LCK manifold with potential is an LCK manifold with a positive Kahler potential on its cover, such that the deck group multiplies the Kahler potential by a constant. A Lee class of an LCK manifold is the cohomology class of the Lee form. We determine the set of Lee classes on LCK manifolds admitting an LCK structure with potential, showing that it is an open half-space in $H^1(M, \mathbb{R})$. For Vaisman manifolds, this theorem was proven in 1994 by Tsukada; we give a new self-contained proof of his result.

A3. L. Ornea, M. Verbitsky, *Algebraic cones of LCK manifolds with potential*, arXiv:2208.05833.

A complex manifold X is called "LCK manifold with potential" if it can be realized as a complex submanifold of a Hopf manifold. Let $\tilde{X}$ its $\mathbb{Z}$-covering, considered as a complex submanifold in $\mathbb{C}^n \backslash 0$. We prove that $\tilde{X}$ is algebraic. We call the manifolds obtained this way **the algebraic cones**, and show that the algebraic structure on $\tilde{X}$ is independent from the choice of X. We give several intrinsic definitions of an algebraic cone, and prove that these definitions are equivalent.

A4. L. Ornea, M. Verbitsky, *A Calabi-Yau theorem for Vaisman manifolds*, arXiv:2206.08808.

A compact complex Hermitian manifold (M, I, ω) is called Vaisman if $d\omega = \omega \wedge \theta$ and the 1-form θ, called the Lee form, is parallel with respect to the Levi-Civita connection. The volume form of M is invariant with respect to the action of the vector field X dual to θ (called the Lee field) and the vector field $I(X)$, called the anti-Lee field. The cohomology class of θ, called the Lee class, plays the same role as the Kähler class in Kähler geometry. We prove that a Vaisman metric is uniquely determined by its volume form and the Lee class, and, conversely, for each Lee class $[\theta]$ and each Lee- and anti-Lee-invariant volume form V, there exists a Vaisman structure with the volume form V and the Lee class $c[\theta]$. This is an analogue of the Calabi-Yau theorem claiming that the Kähler form is uniquely determined by its volume and the cohomology class.

A5. L. Ornea, M. Verbitsky, *Mall bundles and flat connections on Hopf manifolds*, arXiv:2205.14062.

A **Mall bundle** on a Hopf manifold $H = \frac{\mathbb{C}^n \backslash 0}{\mathbb{Z}}$ is a holomorphic vector bundle whose pullback to $\mathbb{C}^n \backslash 0$ is trivial. We define **resonant** and **non-resonant** Mall bundles, generalizing the notion of the resonance in ODE, and prove that a non-resonant Mall bundle always admits a flat holomorphic connection. We use this observation to prove a version of Poincaré-Dulac linearization theorem, showing that any non-resonant invertible holomorphic contraction of $\mathbb{C}^n$ is linear in appropriate holomorphic coordinates. We define the notion of **resonance** in Hopf manifolds, and show that all non-resonant

Hopf manifolds are linear; previously, this result was obtained by Kodaira using the Poincaré-Dulac theorem.

A6. L. Ornea, M. Verbitsky, *Non-linear Hopf manifolds are locally conformally Kaehler*, arXiv:2202.12398.

A Hopf manifold is a quotient of $\mathbb{C}^n \backslash 0$ by the cyclic group generated by a holomorphic contraction. Hopf manifolds are diffeomorphic to $S^1 \times S^{2n-1}$ and hence do not admit Kähler metrics. It is known that Hopf manifolds defined by linear contractions (called linear Hopf manifolds) have locally conformally Kähler (LCK) metrics. In this paper we prove that the Hopf manifolds defined by non-linear holomorphic contractions admit holomorphic embeddings into linear Hopf manifolds, and, moreover they admit LCK metrics.

A7. L. Ornea, M. Verbitsky, V. Vuletescu, *Do products of compact complex manifolds admit LCK metrics?*, arXiv:2211.08111.

An LCK manifold is a Hermitian manifold which admits a Kähler cover with deck group acting by holomorphic homothetie with respect to the Kähler metric. The product of two LCK manifolds does not have a natural product LCK structure. It is conjectured that a product of two compact complex manifolds is never LCK. We classify all known examples of compact LCK manifolds onto three exclusive classes: LCK with potential, Oeljeklaus-Toma, and those containing a rational curve. In the present paper, we prove that a product of an LCK manifold and an LCK manifold belonging to one of these three classes does not admit an LCK structure.

A8. D. Angella, A. Dubickas, A. Otiman, J. Stelzig, *On metric and cohomological properties of Oeljeklaus-Toma manifolds*, **Publicacions Matemàtiques** (2022), accepted.

We study metric and cohomological properties of Oeljeklaus-Toma manifolds. In particular, we describe the structure of the double complex of differential forms and its Bott-Chern cohomology and we characterize the existence of pluriclosed (aka SKT) metrics in number-theoretic and cohomological terms. Moreover, we prove they do not admit any Hermitian metric ω such that $\partial\overline{\partial}\omega^k = 0$, for $2 \leq k \leq n-2$ and we give explicit formulas for the Dolbeault cohomology of Oeljeklaus-Toma manifolds admitting pluriclosed metrics.

A9. N. Istrati, A. Otiman, *Bott-Chern cohomology of compact Vaisman manifolds*, **Transactions of the American Mathematical Society** (2022), accepted.

We give an explicit description of the Bott-Chern cohomology groups of a compact Vaisman manifold in terms of the basic cohomology. We infer that the Bott-Chern numbers and the Dolbeault numbers of a Vaisman manifold determine each other. On the other hand, we show that the cohomological invariants Δ^k introduced by Angella-Tomassini are unbounded for Vaisman manifolds. Finally, we give a cohomological characterization of the Dolbeault and Bott-Chern formality for Vaisman metrics.

A10. M.A. Aprodu, *Pseudo V-harmonic morphisms*, preprint 2022.

We introduce the notion of V-minimality, for V a smooth vector field on a Riemannian manifold. This is a natural extension of the classical notion of minimality.

To emphasis the utility of this notion we present the generalization of pseudo harmonic morphisms. Specifically, we prove that a PHH submersion is V-harmonic if and only if it has minimal fibres and a PHH V-harmonic submersion pulls back complex submanifolds to V minimal submanifolds.

A11. S. Deaconu, V. Vuletescu, *On locally conformally Kähler metrics on Oeljeklaus-Toma manifolds*, **Manuscripta Mathematica** (2022), accepted.
https://doi.org/10.1007/s00229-022-01403-0

We show that Oeljeklaus-Toma manifolds $X(K,U)$ where K is a number field of signature (s,t) such that $s \geq 1, t \geq 2$ and $s \geq 2t$ admit no lck metric. Combined with the earlier results by Oeljeklaus -Toma [An. Inst. Fourier 2005] and Dubickas [New York J. Math. 2014] this completely solves the problem of existence of LCK metrics on Oeljeklaus-Toma manifolds.

A12. V. Slesar, G.-E. Vîlcu, *Vaisman manifolds and transversally Kähler-Einstein metrics*, arXiv:2205.02120v2.

We use the transverse Kähler-Ricci flow on the canonical foliation of a closed Vaisman manifold to deform the Vaisman metric into another Vaisman metric with a transverse Kähler-Einstein structure. We also study the main features of such a manifold. Among other results, using techniques from the theory of parabolic equations, we obtain a direct proof for the short time existence of the solution for transverse Kähler-Ricci flow on Vaisman manifolds, recovering in a particular setting a result of Bedulli, He and Vezzoni [J. Geom. Anal. 28, 697–725 (2018)], but without employing the Molino structure theorem. Moreover, we investigate Einstein-Weyl structures in the setting of Vaisman manifolds and find their relationship with quasi-Einstein metrics. Some examples are also provided to illustrate the main results.

A13. B.-Y. Chen, A.D. Vîlcu, G.-E. Vîlcu, *Classification of graph surfaces induced by weighted-homogeneous functions exhibiting vanishing Gaussian curvature*, **Mediterranean Journal of Mathematics**, 19 (2022), Article number: 162.

Developable surfaces are surfaces in three-dimensional Euclidean space with zero Gaussian curvature. If these surfaces are explicitly defined in the functional form $z = f(x,y)$, then f is nothing but a solution of the homogeneous Monge-Ampère equation. The main aim of this paper is to classify developable surfaces defined as graphs of weighted-homogeneous functions and to apply the result in economic analysis, where the condition of quasi-homogeneity is of particular importance.

A14. S. Deshmukh, H. Al-Sodais, G.-E. Vîlcu, *A note on some remarkable differential equations on a Riemannian manifold*, **Journal of Mathematical Analysis and Applications**, 519(1) (2023), 126778.

The Fischer-Marsden conjecture asserts that an n-dimensional compact manifold admitting a nontrivial solution of the so-called Fischer-Marsden differential equation is necessarily an Einstein space. If this were true, then a classical theorem of Obata would imply that the underlying manifold is either a standard sphere or a Ricci flat space. Although counterexamples to this conjecture have been found by Kobayashi and Lafontaine, it has recently been proved by Cernea and Guan that the Fischer-Marsden conjecture holds, provided that the space of nonconstant solutions of the Fischer-Marsden equation is of dimension at least n, the authors actually proving that in this case (M, g) is nothing but a standard sphere. The main aim of this article is to show that any compact Riemannian manifold of positive Ricci curvature that admits a nontrivial concircular vector field with the potential function satisfying the Fischer-Marsden equation must be isometric to a standard sphere and the converse is also valid. Moreover, we prove that the existence of a nontrivial solution to another remarkable differential equation on Riemannian manifolds, namely the stationary Schrödinger equation, it also leads to a characterization of the sphere, provided that some pinching conditions are satisfied.

A15. S.K. Chaubey, G.-E. Vîlcu, *Gradient Ricci solitons and Fischer–Marsden equation on cosymplectic manifolds*, **Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemática**, 116 (2022), Article number: 186.

In this paper, we investigate the existence of non-trivial solutions for Fischer-Marsden equation within the framework of $(2n + 1)$-dimensional cosymplectic manifolds. It is shown that the existence of such a solution forces the metric to be a gradient η-Ricci soliton. We also explore the geometrical properties of gradient Ricci solitons on η-Einstein cosymplectic manifolds and discuss some illustrative examples in this context.

A16. J.W. Lee, C.W. Lee, B. Şahin, G.-E. Vîlcu, *Chen-Ricci inequalities for Riemannian maps and their applications*, in: *Differential Geometry and Global Analysis: In Honor of Tadashi Nagano*, Editors: B.-Y. Chen et al., **Contemporary Mathematics**, AMS, vol. 777, 2022, 137–152.

Riemannian maps between Riemannian manifolds, originally introduced by A.E. Fischer in [Contemp. Math. 132 (1992), 331–366], provide an excellent tool for comparing the geometric structures of the source and target manifolds. Isometric immersions and Riemannian submersions are particular examples of such maps. In this work, we first prove a geometric inequality for Riemannian maps having a real space form as a target manifold. Applying it to the particular case of Riemannian submanifolds, we recover a classical result, obtained by B.-Y. Chen in [Glasgow Math. J. 41 (1999), 33–41], which nowadays is known as the Chen-Ricci inequality. Moreover, we extend this inequality in case of Riemannian maps with a complex space form as a target manifold, improving a geometric inequality stated by B.-Y. Chen in [Arch. Math. (Basel) 74 (2000), 154–160].

A17. C. Gherghe, G.-E. Vîlcu, *Harmonic maps on locally conformal almost cosymplectic manifolds*, preprint 2022.

We investigate harmonic maps on almost contact metric manifolds which are locally conformal to almost cosymplectic manifolds. We obtain the necessary and sufficient conditions for the holomorphy to imply harmonicity and then we find obstructions to the existence of non-constant pluriharmonic maps. We also establish some results on the stability of the identity map on a locally conformal almost cosymplectic manifold of pointwise constant ϕ-holomorphic sectional curvature.

2 Progress Summary

The 2022 stage was completed with the full fulfillment of the assumed objectives (O1-O3), realizing a number of 17 scientific articles (A1-A17). The dissemination of the research results was achieved by participating with invited presentations at international conferences and in departmental seminars, as follows.

- Invited talks at international conferences

  1. L. Ornea: *Lee classes of LCK manifolds with potential*, Recent advances in complex and symplectic geometry (Online), January 10-12, 2022, Parma.
  2. L. Ornea: *A Calabi-Yau theorem for Vaisman manifolds*, Kaehler and non-Kaehler geometry: New developments and interactions. June 21-23, 2022, Aarhus.
  3. M. Stanciu: *Modifications of locally conformally Kähler spaces*, Workshop for Young Researchers in Geometry, May 19 – May 20, 2022, Bucharest.
  4. G.-E. Vîlcu: *Killing forms on Sasaki-Einstein spaces and transverse Kähler-Ricci flow*, Workshop on Differential Geometry, Sungkyunkwan University, Suwon, South Korea, 8-9 August 2022.

- Invited departmental talks

  1. L. Ornea: *Coni algebrici su varietà LCK con potenziale*, 01.12.2022, University of Rome 3, Italy.
  2. A. Otiman: *New constructions in non-Kähler toric geometry*, martie 2022, Chalmers University, Goteborg, Sweden.
  3. A. Otiman: *Cohomological and metric properties of Oeljeklaus Toma manifolds*, May 2022, Paris Orsay, France.
  4. M. Stanciu: *Modifications of locally conformally Kähler spaces*, September 2022, Aarhus University, Denmark.
  5. V. Slesar: *Vaisman manifolds, transverse Kähler-Ricci flow and Einstein-Weyl structures"*, Differential Geometry Seminar, University of Torino, 22.11.2021.

From the above, we deduce the following result indicators:

- Articles published in ISI indexed journals: 4
- Articles accepted in ISI indexed journals: 4
- Articles under evaluation in ISI indexed journals: 8

- Presentations at conferences: 9
- Book chapters: 1

The mobilities settled from the 2022 stage budget were the following.

1. L. Ornea: 03.03-09.03.2022, mobility in Italy, at the University of Rome 3, for scientific collaboration with Prof. Massimiliano Pontecorvo on topics of common interest from locally conformal Kähler geometry.
2. V. Marchidanu: 22.05.2022-13.06.2022, mobility in Brazil, at the Instituto Nacional de Matemática Pura e Aplicada (IMPA) Rio de Janeiro, for scientific collaboration with Prof. Misha Verbitsky on topics of common interest from locally conformal Kähler geometry.
3. M. Stanciu: 26.08.2022-02.09.2022, mobility in Danemarca, at the Aarhus University, for an invited lecture at the Geometry Seminar of the Mathematics Department of Aarhus University and scientific collaboration with the LCK geometry specialists in the department.
4. L. Ornea: 21.01.2023-05.02.2023, mobility in Italy at "Centro Internazionale per la Ricerca Matematica Trento" for research in order to fulfill the objectives of the grant and participation as a main speaker at the international conference "Cohomology of Complex Manifolds and Special Structures - III".
5. V. Vuletescu: 21.01.2023-28.01.2023, mobility in Italy at "Centro Internazionale per la Ricerca Matematica Trento" for participating with an invited talk at the international conference "Cohomology of Complex Manifolds and Special Structures - III".

3 Summary of the stage

In its 2022 stage, entitled *Structure of the LCK manifolds and spaces*, financing contract no. 30/2021 aiming to obtain new geometric and topological properties of locally conformal Kähler manifolds, in accordance with the stated objectives.

The activities carried out consisted in:

- documentation and information,
- analysis and conception of solutions for realization,
- elaboration of scientific papers and
- participation in scientific events for the dissemination of the obtained results.

All activities were 100% completed by the deadline and according to the budget allocated by the contract and the objectives were fully achieved.

The output was:

- A number of 17 scientific papers (which is more than four times the planned number).
- 9 invited talks at international conferences and departmental seminars.

Project manager,

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