Scientific report on the implementation of the project PN-III-P4-ID-PCE-2020-0025 during 2021

1 Synopsis

In its 2021 stage, entitled *Classification and counting on LCK manifolds*, financing contract no. 30/2021 aimed at the realization of scientific research activities through which to obtain new properties on the geometry and topology of locally conformal Kähler manifolds, in accordance with the stated objectives.

These specific objectives in the 2021 stage of the project were the following: counting the elliptic curves on compact Vaisman manifolds, description of all possible values of the cohomology class of Lee forms and the non-existence of LCK metrics on certain Oeljeklaus-Toma manifolds.

The activities carried out consisted in: documentation and information, analysis and conception of solutions for realization, elaboration of articles and participation in scientific events for dissemination of the obtained results. All activities were completed in proportion of 100% by the deadline and according to the budget allocated by contract.

The research undertaken by the project team in the 2021 stage was materialized through a number of 6 scientific articles, which represents the double of the planned number of articles. The dissemination of the results was achieved by participating with 18 invited talks at international conferences and departmental seminars.

2 Scientific and technical description

2.1 Published papers and preprints

During this period 6 scientific papers were elaborated, 2 among them being published in ISI-ranked journals and 4 papers are under review. The content of these 6 papers, that cover completely the objectives proposed for this period, can be synthesized as follows:

1. L. Ornea, M. Verbitsky: *Closed orbits of Reeb fields on Sasakian manifolds and elliptic curves on Vaisman manifolds*, **Mathematische Zeitschrift** 299 (2021), 2287–2296.

A compact complex manifold V is called Vaisman if it admits a Hermitian metric which is conformal to a Kähler one, and a non-isometric conformal action by $\mathbb{C}$. It is called quasi-regular if the $\mathbb{C}$-action has closed orbits. In this case the corresponding

leaf space is a projective orbifold, called the quasi-regular quotient of V. It is known that the set of all quasi-regular Vaisman complex structures is dense in the appropriate deformation space. We count the number of closed elliptic curves on a Vaisman manifold, proving that their number is either infinite or equal to the sum of all Betti numbers of a Kähler orbifold obtained as a quasi-regular quotient of V. We also give a new proof of a result by Rukimbira showing that the number of Reeb orbits on a Sasakian manifold M is either infinite or equal to the sum of all Betti numbers of a Kähler orbifold obtained as an S^1-quotient of M.

2. C.W. Lee, J.W. Lee, G.-E. Vîlcu, *Classification of Casorati ideal Legendrian submanifolds in Sasakian space forms II*, **Journal of Geometry and Physics** 171 (2022), 104410.

We prove two optimal inequalities involving the generalized Casorati curvatures $\delta_C(r; n-1)$ and $\widehat{\delta_C}(r; n-1)$ of n-dimensional Legendrian submanifolds in Sasakian space forms, where r is any real number such that $0 < r < n(n-1)$ or $r > n(n-1)$, respectively. We classify those submanifolds for which the equality cases of both inequalities hold and provide examples of submanifolds that are ideal for any given r.

3. L. Ornea, M. Verbitsky, *Compact homogeneous locally conformally Kahler manifolds are Vaisman. A new proof*, arXiv:2110.12361.

An LCK manifold with potential is a complex manifold with a Kähler potential on its cover, such that any deck transformation multiplies the Kähler potential by a constant multiplier. We prove that any homogeneous LCK manifold admits a metric with LCK potential. This is used to give a new proof that any compact homogeneous LCK manifold is Vaisman.

4. T. Albu, S. Dăscălescu, *Free objects and coproducts in categories of posets and lattices*, preprint 2021.

A classical result in universal algebra says that for any type τ of an algebraic system, given by a set of operators and a set of identities, there exists a free τ-algebra over an arbitrary set, i.e., the forgetful functor from the category $\mathbf{Alg}_\tau$ of algebras of type τ to the category **Set** of sets has a left adjoint. We consider the category **Lat** of lattices and the category **bLat** of bounded lattices, i.e., lattices with a least element 0 and a greatest element 1. In the first one, morphisms are the mappings commuting with finite meets and joins, while in the latter one, morphisms also preserve 0 and 1. Both these categories are of the type $\mathbf{Alg}_\tau$, so the forgetful functors from these categories to **Set** have left adjoints. On the other hand, colimits, in particular coproducts, exist in any $\mathbf{Alg}_\tau$.

Our aim is to discuss the existence of free objects and coproducts in the full subcategory $\overline{\mathbf{bLat}}$ of **Lat** whose objects are all bounded lattices. The morphisms in $\overline{\mathbf{bLat}}$ are just lattice morphisms, not necessarily preserving 0 and 1. We also consider similar questions for posets. If **Pos** is the category of posets, with monotone mappings as morphisms, **bPos** is the category of bounded posets, i.e., posets with a least element 0 and a greatest element 1, and morphisms preserving 0 and 1, and $\overline{\mathbf{bPos}}$ is the full subcategory of **Pos** whose objects are the bounded posets, we discuss the existence of free objects and coproducts in $\overline{\mathbf{bPos}}$.

We show that a free object over any set with at least two elements does not exist in $\overline{\mathbf{bPos}}$, and a free object over a set X exists in $\overline{\mathbf{bLat}}$ if and only if X is finite.

We also prove that $\overline{\mathbf{bPos}}$ does not have any coproducts, while $\overline{\mathbf{bLat}}$ has finite coproducts, but it does not have any infinite coproduct. On the other hand, as we explained above, coproducts exist in **Lat**; their existence can be obtained as an application of the adjoint functor theorem. We briefly include a construction of such coproducts, since they are related to our result.

5. D. Angella, M. Parton, V. Vuletescu, *On locally conformally Kähler threefolds with algebraic dimension two* - preprint 2021.

The paper is part of an attempt of understanding non-Moishezon non-Kähler threefolds. We start by looking at the simplest case: threefolds X with algebraic dimension $a(X) = 2$. In this case, it is classically known that they are bimeromorphic to elliptic fibrations over projective surfaces, that is, there exists a smooth bimeromorphic threefold X^* and a surjective holomorphic map $f\colon X^* \to B$ whose general fibres are smooth elliptic curves. In fact, X^* is an *algebraic reduction* of X. The main goal of the paper is to give a description of X^*, and then retrieve information about X.

By generalizing what happens on non-Kähler surfaces S of algebraic dimension $a(S) = 1$ (see Proposition 3.17 of [V. Brînzănescu, *Holomorphic vector bundles over compact complex surfaces*, *Lecture Notes in Mathematics*, vol. 1624, Springer-Verlag, Berlin, 1996]), we prove that, under mild assumptions that we now describe, the algebraic reduction is a *quasi-bundle*, namely, the fibres with the reduced structure are smooth elliptic isomorphic curves. The main idea of our strategy is inspired by the Lefschetz hyperplane theorems in algebraic geometry. More exactly, we consider divisors H on B, and look at their preimages $S_H := f^{-1}(H) \subset X$ taking advantage of the known results for compact complex non-Kähler surfaces.

6. O. Preda, M. Stanciu: *Vaisman theorem for LCK spaces* - preprint 2021.

Vaisman's theorem for locally conformally Kähler (lcK) compact manifolds states that any lcK metric on a compact complex manifold which admits a Kähler metric is, in fact, globally conformally Kähler (gcK). In this paper, we extend this theorem to compact complex spaces with singularities.

2.2 Invited talks at international conferences

1. L. Ornea: *Elliptic curves on Vaisman manifolds*, Cohomology of Complex Manifolds and Special Structures - II, 5-9.07, 2021, Levico Terme, Italia.

2. L. Ornea: *Locally conformally Kähler geometry. An overview*, Locally Conformal Symplectic Manifolds: Interactions and Applications, Banff, Canada, November 7-12, 2021.

3. A. Otiman: *Special non-Kähler metrics on solvmanifolds* (online), Complex Geometry and Lie Groups VI, February 2021, Niigata, Japan.

4. A. Otiman: *Special non-Kähler metrics on solvmanifolds*, Workshop for Young Researchers in Mathematics, Bucureşti, 20-21.05.2021.

5. A. Otiman: *Geometry of Kato manifolds* (online), Minisymposium "Topics in complex and quaternionic geometry", sattelite event of the European Congress in Mathematics, June 2021, Portoroz, Slovenia.

6. A. Otiman: *Hermitian geometry of Oeljeklaus-Toma manifolds*, Cohomology of Complex Manifolds and Special Structures II, July 2021, Levico Terme, Italy.

7. A. Otiman: *Topics in locally conformally Kähler geometry* (online), 6th Geometry-Topology Summer School, August 2021, Feza Gürsey Institute, Turcia.

8. A. Otiman: *New Constructions in non-Kähler toric geometry* (online), PRIN seminar, October 2021, University of Florence.

9. A. Otiman: *Oeljeklaus-Toma manifolds*, Workshop on Balanced metrics and Monge Ampere masses, November 2021, University of Parma.

10. A. Otiman: *Hermitian geometry of Oeljeklaus-Toma manifolds* (online), Workshop on locally conformally symplectic manifolds: interactions and applications, Banff, Canada, November 2021.

11. A. Otiman: *New Constructions in non-Kähler toric geometry* (online), "Recent Developments in Complex Geometry and Geometric Analysis", Canadian Mathematical Society Winter Meeting, December 2021.

12. M. Stanciu: *Coverings of locally conformally Kähler complex spaces*, Workshop for Young Researchers in Mathematics, Bucureşti, 20-21.05.2021.

13. G.-E. Vîlcu: *Curvature invariants, optimal inequalities and ideal submanifolds in space forms*, 18th International Geometry Symposium, Inonu University, Malatya, 12-13 July 2021.

2.3 Invited departmental talks

1. A. Otiman: *Toric Kato manifolds*, Almost Complex Geometry Seminar, City University of New York, March 2021.

2. A. Otiman: *Variational problems in conformal geometry*, Geometry Seminar, Vanderbilt University, March 2021.

3. A. Otiman: *Old and new constructions in non-Kähler geometry* (online), Aarhus University, May 2021.

4. A. Otiman: *Toric Kato manifolds* (online), Séminaire Géométrie Complexe, Université Aix-Marseille, December 2021.

5. M. Stanciu: *Vaisman's theorem for lcK spaces with singularities*, Geometry Seminar, University of Florence, 09.11.2021.

2.4 Mobilities

1. L. Ornea: 27.05-26.06.2021, mobility in Italy, at the University of Rome 3, for scientific collaboration with Prof. Massimiliano Pontecorvo on topics of common interest from locally conformal Kähler geometry.

2. C. Ciulică: 05.11-15.11.2021, mobility at the Università degli Studi Firenze, Italy, for scientific collaboration with Prof. Daniele Angella on topics of common interest from locally conformal Kähler geometry (related to his PhD thesis, connected with the grant's theme).

3. M. Stanciu: 05.11-12.11.2021, mobility at the Università degli Studi Firenze, Italy, for scientific collaboration with Prof. Daniele Angella on topics of common interest from locally conformal Kähler geometry and for an invited talk at the "Geometry Seminar" of the university.

4. V. Marchidanu: 25.09.2021-17.10.2021, mobility in France, at the University of Strasbourg, for scientific collaboration with Prof. Alexandru Oancea on topics of common interest from (locally conformal) symplectic geometry (related to his Master thesis, connected with the grant's theme).

Project manager,

Prof. dr. Liviu Ornea